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LETTER TO THE EDITOR

The analogue of the Bäcklund transformation for integrable many-body systems

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Abstract. Canonical transformations analogous to the Bäcklund transformation are discussed. Explicit formulae are given for many-body systems of particles interacting in one dimension.

There are three characteristic properties common to all soliton equations: their Lax representation and solvability by the inverse scattering transform, their complete integrability and the existence of the Bäcklund transformation. Similar properties hold true for the finite-dimensional counterparts of the soliton equations, i.e. systems of many particles (N) interacting in a line by the following two-body potentials:

$$V(x) = \mathrm{e}^x, \qquad \text{for nearest-neighbour interactions,} \tag{1}$$

$$\begin{aligned} &V(x) = \rho(x) \qquad \text{(special cases } 1/x^2, \coth^2 x, \cot^2 x\text{) or} \\ &V(x) = 1/x^2 + \omega^2 x^2 \qquad \text{for interaction with each other.} \end{aligned} \tag{2}$$

Hamiltonian equations describing the motion of those systems also admit Lax representations and have N independent commuting first integrals, thus ensuring complete integrability (Flaschka 1974, Moser 1975, Calogero 1975). However, the analogue of the Bäcklund transformation has been to some extent unknown. Here we give adequate formulae for all the potentials written above.

First let us explain what is understood by a Bäcklund transformation for a mechanical system. In the case of soliton equations it is a system of two lower order (than that of the original equation) partial differential equations having the following characteristic features:

(i) the conditions of compatibility reduce to the dynamical equation,
(ii) they generalise contact transformation,
(iii) they provide (via the permutability theorem) the algebraic construction of new solutions.

The important limitation of the Bäcklund transformation is that only a narrow class of new solutions can be constructed using it.

For a mechanical system there is a unique possibility of satisfying the aforementioned conditions if the transformation between the old and new variables y, x takes the form $\dot{x} = f(x, y)$, $\dot{y} = g(x, y)$ and expressions for $\ddot{x}$, $\ddot{y}$ reduce to the equations of

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motion. The first example is the Kac–Van Moerbecke (1975) system of equations:

$$\dot{x}_k = \exp(x_k - x_{k+1}) + \exp(x_{k-1} - x_k) \qquad k = 1, \ldots, N \qquad x_0 = -\infty, x_{N+1} = +\infty \tag{3}$$

which can be differentiated to give the system of two disjointed Toda sublattices

$$\ddot{x}_k = \exp(x_{k-2} - x_k) - \exp(x_k - x_{k+2}) \tag{4}$$

corresponding to even and odd values of k. For N even, equations (3) connect two solutions of the ($\frac{1}{2}N$)-particle system and for N odd they set a reduction of the solution for $\frac{1}{2}(N+1)$ particles to $\frac{1}{2}(N-1)$ particles. The next example of the Bäcklund transformation has been found recently by J Gibbons (private communication) for the Calogero (1975) system of N particles interacting with each other by the potential $1/x^2$. That result has been a byproduct of his research on the rational solutions of the Schrödinger equation $\mathrm{i}\psi_t = -\psi_{zz} + U(z)\psi$ with the potential $U(z) = \Sigma_{k=1}^{N} 2/(z-y_k)^2$. His formulae correspond to a particular case of the general result presented below.

The analogue of the Bäcklund transformation for the one-dimensional system of N particles interacting with each other by one of the particular potentials (2) is defined by the expressions

$$-\mathrm{i}\varepsilon\dot{x}_k = -2\sum_{j=1}^{M}{}' \phi(x_k - x_j) + 2\sum_{j=1}^{N} \phi(x_k - y_j) - 2\mathrm{i}\lambda\varepsilon + \omega x_k \tag{5}$$

$$\mathrm{i}\varepsilon\dot{y}_m = 2\sum_{j=1}^{M} \phi(y_m - x_j) - 2\sum_{j=1}^{N}{}' \phi(y_m - y_j) + 2\mathrm{i}\lambda\varepsilon - \omega y_m \tag{6}$$

with the following specification of ϕ functions:

(a) $V(x) = \rho(x)$: $\phi(x) = \zeta(x)$, $M = N$, $\omega = 0$;
(b) $V(x) = 1/x^2$ ($\coth^2 x$, $\cot^2 x$): $\phi(x) = 1/x$ ($\coth x$, $\cot x$), M, N arbitrary, $\omega = 0$;
(c) $V(x) = 1/x^2 + \omega^2 x^2$: $\phi(x) = 1/x$, M, N arbitrary, $\omega \neq 0$;

where $\rho(x)$, $\zeta(x)$ are the Weierstrass functions, $k = 1, \ldots, M$, $m = 1, \ldots, N$, and the values of $\varepsilon = 1$, $\varepsilon = -\mathrm{i}$ correspond to the case of repulsive and attractive force respectively. The prime on the Σ symbol means that the singular term is omitted in the summation.

This system of equations has all characteristic features of the Bäcklund transformation:

(i) Time differentiation of equations (5) and (6) and substitution of $\dot{x}$, $\dot{y}$ reduce them to the potential motion equations if

(a) $M = N$, $\omega = 0$ and the function $\phi(x)$ satifies the functional equation

$$\begin{aligned} &\phi(x)\phi'(y) - \phi'(x)\phi(y) = \phi(x+y)[\phi'(y) - \phi'(x)] + \Gamma(x) - \Gamma(y) \\ &\phi(-x) = \phi(x) \qquad \Gamma(-x) = -\Gamma(x) \end{aligned} \tag{7}$$

with the most general solution (Choodnovsky and Choodnovsky 1977) given by

$$\phi(x) = \zeta(x) + \gamma x \qquad \text{and} \qquad \Gamma(x) = \tfrac{1}{2}\phi''(x) + \phi(x)\phi'(x).$$

Without any loss of generality of equations (5) and (6) the value $\gamma = 0$ can be taken here. Direct, but tedious, computation confirms that the equations of motion (for

$\varepsilon = 1$) are

$$\ddot{x}_k = -\sum_{j=1}^{N}{}' \rho'(x_k - x_j) \qquad \ddot{y}_m = -\sum_{j=1}^{N}{}' \rho'(y_m - y_j).$$

(b) M, N arbitrary, $\omega = 0$ and the function $\phi(x)$ satisfies the following variant of the functional equation (7):

$$\phi(x)\phi'(y) - \phi'(x)\phi(y) = \phi(x+y)[\phi'(y) - \phi'(x)]$$

with the most general solution (Choodnovsky and Choodnovsky 1977): $1/x$, $\coth x$, $\cot x$. The case $\phi(x) = 1/x$ has been found by J Gibbons (private communication).

(c) M, N arbitrary, $\omega \neq 0$ and the function $\phi(x) = 1/x$. Then by direct computation one verifies that

$$\ddot{x}_k = \sum_{j=1}^{M}{}' 1/(x_k - x_j)^3 - \omega x_k \qquad \ddot{y}_m = \sum_{j=1}^{N}{}' 1/(y_m - y_j)^3 - \omega y_m. \tag{8}$$

(ii) The transformation of (5) and (6) is canonical because $\dot{x}_k = \partial F/\partial x_k$, $\dot{y}_k = -\partial F/\partial y_k$ with generating function

$$F(x, y) = \frac{\mathrm{i}}{\varepsilon} \ln \frac{\prod_k^N \prod_m^N \sigma^2(x_k - y_m)}{\prod_{j>k}^N \prod_k^N \sigma^2(x_j - x_k) \prod_{m>n}^N \prod_n^N \sigma^2(y_m - y_n)} + 2\lambda\left(\sum_{k=1}^{N} x_k - \sum_{m=1}^{N} y_m\right) + \frac{\mathrm{i}\omega\varepsilon}{2}\left(\sum_{k=1}^{N} x_k^2 - \sum_{m=1}^{N} y_m^2\right) \tag{9}$$

where $\sigma(x)$ is also a Weierstrass function. Note that it is a very particular time-independent canonical transformation which conserves not only the Hamiltonian character of the equations of motion but also the algebraical form of the Hamiltonian. Also for N even the Kac–Van Moerbecke equations (3) define a canonical transformation with the generating function

$$F(x_1, \ldots, x_N) = \sum_{k=1}^{N-1} (-1)^k \exp(x_k - x_{k+1}).$$

(iii) The transformation defined by equations (5) and (6) is a system of $N+M$ algebraic equations for $2M$ quantities x_i, $\dot{x}_i$ once the solution y_j, $\dot{y}_j$ of the equations of motion is given. Thus the question naturally arises of the existence and uniqueness of the solution to the underdetermined (for $M < N$) system of equations. The answer comes out of the analysis of the Schrödinger equation

$$\mathrm{i}\psi_t = -\psi_{zz} + U(z)\psi. \tag{10}$$

In the simplest case of the potential $U(z) = \sum_{j=1}^{N} 2/(z - y_j)^2$ one looks for the rational solution

$$\psi(z, t, \lambda) = \frac{\sum_{i=1}^{M} (z - x_i(t))}{\sum_{j=1}^{N} (z - y_j(t))} \exp[\mathrm{i}\varepsilon(\lambda z - \lambda^2 t)] \tag{11}$$

which, for $M = N$, can also be decomposed into the sum of simple fractions

$$\psi(z, t, \lambda) = \left(1 + \sum_{j=1}^{N} b_j/(z - y_j)\right) \exp[\mathrm{i}\varepsilon(\lambda z - \lambda^2 t)]. \tag{12}$$

The substitution of ψ into (10) and computation of residues provides the two following systems of equations

$$-\mathrm{i}\varepsilon\dot{x}_k = -2\sum_{j=1}^{N}{}' 1/(x_k - x_j) + 2\sum_{j=1}^{N} 1/(x_k - y_j) - 2\lambda\mathrm{i}\varepsilon \tag{13}$$

$$\mathrm{i}\varepsilon\dot{y}_m = 2\sum_{j=1}^{N} 1/(y_m - x_j) - 2\sum_{j=1}^{N}{}' 1/(y_m - y_j) + 2\lambda\mathrm{i}\varepsilon \tag{14}$$

and

$$\mathrm{i}\varepsilon\dot{y}_k b_k - 2\mathrm{i}\lambda\varepsilon b_k = \sum_{j=1}^{N}{}' 2b_j/(y_k - y_j) + 2 \tag{15}$$

$$\mathrm{i}\varepsilon\dot{b}_k = 2\left(b_k \sum_{j=1}^{N}{}' 1/(y_k - y_j)^2 - \sum_{j=1}^{N}{}' b_j/(y_k - y_j)^2\right) \tag{16}$$

which are equivalent to either equation (10) with *ansatz* (11) or equation (12). Thus, for $M = N$, the Bäcklund transformation defined by (13) and (14) is, in fact, equivalent to the Lax form of the Calogero (1975) system. It can be seen more clearly if equations (15) and (16) are put into matrix notation:

$$Lb = 2\lambda\varepsilon b - 2\mathrm{i}v \tag{17}$$

$$\varepsilon\dot{b} = Ab \tag{18}$$

where

$$L_{jk} = \delta_{jk}y_j + (1 - \delta_{jk})\frac{2\mathrm{i}}{y_j - y_k}$$

$$A_{jk} = \delta_{jk}\left(-2\mathrm{i}\sum_{m=1}^{N}{}' \frac{1}{(y_j - y_m)^2}\right) + \frac{1 - \delta_{jk}}{(y_j - y_k)^2}.$$

v is a column vector with unit components and b is also a column vector with components b_m, $m = 1, \ldots, N$. If y_j, $\dot{y}_j$ is a solution of the equations of motion, then equation (17) is a system of linear equations for b_m, which has a unique solution, provided 2λ is not an eigenvalue of L. The positions x_k can now be determined as the roots of the Nth order polynomial with coefficients dependent on y_j and b_m. If all roots are distinct the solution is unique. Note that the assumption of y_j, $\dot{y}_j$ satisfying the equations of motion is compatible with equations (17) and (18). Indeed, the time differentiation of (17) and substitution of (18) gives the Lax form

$$\dot{L} = [A, L]b$$

of the equations of motion on y_j, $\dot{y}_j$.

The transformation (13), (14) determined a one-parameter family $x_j(t, \lambda)$, $\dot{x}_j(t, \lambda)$ of Bäcklund transforms. Unfortunately for the repulsive ($\varepsilon = 1$) case it takes complex values. Only in the attractive ($\varepsilon = -\mathrm{i}$) case, if at some time x_j, $\dot{x}_j$ are real, do they remain real later. However, here there are difficulties with the singularities of the kinetic energy in finite time, and hence the transformation is well defined within the time intervals between collapses of particles.

For $M = N - 1$ the system of equations (13), (14) defines the reduction of the solution for N particles to the solution for $N - 1$ particles. Here again y_j, $\dot{y}_j$ are specified arbitrarily and there are $2N - 2$ variables x_i, $\dot{x}_i$ to determine from $2N - 1$

equations. Therefore the value of λ also has to be determined from the equations. In fact, by virtue of the decomposition

$$\frac{\Pi_{i=1}^{N-1}(z-x_i)}{\Pi_{j=1}^{N}(z-y_j)}=\sum_{j=1}^{N}\frac{b_j}{z-y_j} \tag{19}$$

the system of (13) and (14) is equivalent to the matrix equations

$$Lb=2\lambda\varepsilon b \qquad \varepsilon b=Ab.$$

The non-trivial solution for b exists only for a particular λ—the eigenvalues of the matrix L. Positions of the particles x_i are the roots of the polynomial determined from (19). Thus for every eigenvalue $2\lambda_k$, $k=1,\ldots,N$, there exists a unique solution of equations (13) and (14).

The case $M=0$ for $\varepsilon=-\mathrm{i}$ is known (Choodnovsky and Choodnovsky 1977) as having a Lax representation and embeddable into the Hamiltonian system with the potential $-1/x^2$. It describes a very special evolution with velocities algebraically dependent on the positions of the particles. It is remarkable that such a condition propagates in time. Whether the general system of (5) and (6) also has a Lax representation is unknown as yet.

The argument presented above works also for other cases listed in equations (5) and (6). It is necessary to consider the Schrödinger equation (10) with the potential $U(z)=\Sigma_{j=1}^{N}1/\sin^2(z-y_j)$ or $U(z)=\Sigma_{j=1}^{N}\rho(z-y_j)$. Here the use of the analogous *ansatz*

$$\psi\exp[-\mathrm{i}\varepsilon(\lambda x-\lambda^2 t)]=\frac{\Pi_{i=1}^{N}\sigma(z-x_i)}{\Pi_{j=1}^{N}\sigma(z-y_j)} \qquad \left(\text{or } \frac{\Pi_{i=1}^{N}\sin(z-x_i)}{\Pi_{j=1}^{N}\sin(z-y_j)}\right)$$

and of the decomposition

$$\frac{\Pi_{i=1}^{N}\sigma(z-x_i)}{\Pi_{j=1}^{N}\sigma(z-y_j)}=b_0+\sum_{k=1}^{N}b_k\zeta(z-y_k) \qquad \text{for } \sum_{i=1}^{N}x_i=\sum_{j=1}^{N}y_j$$

also makes the Bäcklund transformation equations (5) and (6) equivalent to the Lax representation of the equations of motion.

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